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TRANSVERSE OSCILLATIONS IN A PARTIALLY COMPENSATED ELECTRON BEAM

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Discussion of the motion of not only electrons but also ions of a beam is significant in connection with the investigation of the problem of the stability of a quasisteady quasirelativistic beam. Instabilities of a beam partially compensated with respect to deflection (instabilities of the "snake" type) are discussed in [1]. A model of two filaments formed by electrons and ions of a beam which can shift relative to each other was used. The structure of the beams in phase space is not important for the problems investigated in [1] — the transverse oscillations are analyzed from the motion of axial particles.

The stability of an electron—ion beam relative to axisymmetric perturbations of the radii of the electron and ion components is discussed in this paper. We will assume that both the electrons and the ions of the beam are characterized by a nonzero emittance.

1. It is necessary in connection with the description of the beam particles with the help of a distribution function in the nonsteady case to find an integral of the motion which is not a consequence of the uniformity of the system, which is possible in the paraxial approximation [2, 3].

We will seek the electron distribution function in the form

$$f_e = \varkappa\delta(I - I_{0e})\ \delta(\beta_z - \beta_0), \tag{1.1}$$

where $\varkappa$ is a normalization constant; $\beta_z = v_z/c$; v_z, longitudinal velocity of the electrons; c, speed of light; and I, a functional which depends on the transverse coordinates and velocities.

Satsifaction of the condition $J \ll \gamma\beta_0 mc^3/e$, where J is the total current of the beam, e and m are the charge and mass of the electron, and γ is a relativistic factor, is necessary for the validity of (1.1). The quantity β_z is an approximate integral of the motion which is a consequence of conservation of the z-component of the generalized momentum.

One can represent the function I in the case of an axisymmetric beam under discussion in the form

$$I = A_e(t)\left[\left(\dot{r} - \frac{\dot{A}_e r}{2A_e}\right)^2 + \frac{C_{0e}^2}{r^2}\right] + \frac{E_{0e}^2}{A_e(t)}r^2, \tag{1.2}$$

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where E_{oe} and C_{oe} are constants; C_{oe} has the meaning of the momentum of the transverse velocity $\mathbf{v}_\perp$ relative to the beam axis, i.e., $C_{oe} = rv_\perp \sin\varphi$, and φ is the angle between $\mathbf{v}_\perp$ and $\mathbf{r}_\perp$. The constant E_{oe} is defined in terms of the beam emittance. Using Maxwell's equations div $\mathbf{E} = 4\pi\rho$ and rot $\mathbf{H} = 4\pi j/c$, one can write the equation of motion of an electron in the form

$$\ddot{r} + \omega_e^2(t)\,r - \frac{C_{0e}^2}{r^3} = 0, \tag{1.3}$$

on the assumption of constancy of the electron and ion densities, where $\omega_e^2 = 2\pi e^2/m\gamma(n_i - (n_e/\gamma^2)) + \omega_{He}^2/4$, n_e and n_i are the electron and ion densities of the beam, $\omega_{He} = eH/\gamma mc$, and H is the external axial magnetic field.

Since (by stipulation) I is an integral of the motion, then $dI/dt = 0$, which, together with (1.3), gives an equation for $A_e(t)$:

$$\frac{\ddot{A}_e}{2A_e} = -\omega_e^2 + \frac{E_{0e}^2}{A_e^2} + \frac{\dot{A}_e^2}{4A_e^2}. \tag{1.4}$$

Integrating the electron distribution function over the velocities, we obtain

$$n_e = \int f_e v_\perp dv_\perp d\varphi d\beta_z = \frac{\pi\varkappa}{A_e}\sigma(R_e - r),$$

where

$$\sigma(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geqslant 0; \end{cases}$$

$$R_e^2 = \frac{A_e(t)\,I_{0e}}{E_{0e}^2}. \tag{1.5}$$

Equation (1.5) permits introducing the beam radius $R_e(t)$ into (1.4) in place of $A_e(t)$.

One can describe the ion component of the beam with the help of an analogous model; we assume that one can neglect the longitudinal velocity of the ions, i.e., $v_{iz} \equiv 0$. One can derive an equation for $A_i(t)$ of the form (1.4) with the frequency $\omega_i^2(t) = 2\pi e^2/M(n_e - n_i) + \omega_{Hi}^2/4$, where $\omega_{Hi} = eH/Mc$ and M is the mass of the ion.

Expressing the electron and ion densities in terms of the radii $n_e = N_e/\pi R_e^2$ and $n_i = N_i/\pi R_i^2$, where N_e and N_i are the constant running densities of electrons and ions, we obtain the following equations for the radii:

$$\begin{aligned} &\ddot{R}_e + \frac{2e^2}{m\gamma}\left(N_i\frac{R_e}{R_i^2} - \frac{N_e}{\gamma^2 R_e}\right) + \frac{\omega_{He}^2}{4}R_e - \frac{2e^2}{m\gamma}\frac{\left(N_i - \frac{N_e}{\gamma^2}\right)R_{0e}^2}{R_e^3} = 0, \\ &\ddot{R}_i + \frac{2e^2}{M}\left(N_e\frac{R_i}{R_e^2} - \frac{N_i}{R_i}\right) + \frac{\omega_{Hi}^2}{4}R_i - \frac{2e^2}{M}\frac{(N_e - N_i)\,R_{0i}^2}{R_i^3} = 0. \end{aligned} \tag{1.6}$$

The constants R_{oe} and R_{oi}, which can be expressed with the help of the constants I_o and E_o, characterize the emittance (i.e., the phase volume in the coordinates r and r') of the electron and ion components of the beam. The factors $(2e^2/m\gamma)(N_i - N_e/\gamma^2)$ and $(2e^2/M)(N_e - N_i)$ are introduced for convenience. We note that equations of the form (1.6) (with an emittance different from zero) can be introduced in different ways (e.g., see [4], in which averaging of the equations of motion over an ensemble is used, and [5], in which a hydrodynamic description is used).

System (1.6) has meaning, strictly speaking, if $R_i(t) \equiv R_e(t)$, since the equations of motion of ions and electrons of form (1.3) are valid only in the case of a uniform distribution of the charge density in the region of motion of the particles.

Satisfaction of the conditions

$$\frac{1}{m\gamma}\left(N_i - \frac{N_e}{\gamma^2}\right) = \frac{1}{M}(N_e - N_i),\ R_{0e} = R_{0i},\ H = 0 \tag{1.7}$$

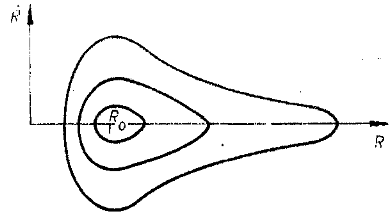

Fig. 1

is necessary for the identity of Eqs. (1.6). It follows from this that

$$\alpha = \frac{N_i}{N_e} = \frac{M + m\gamma^3}{\gamma^2 (M + m\gamma)}.$$

The ratio of the densities cannot be arbitrary and differs little from the condition of force-free motion of the electrons for the case of a comparatively long-lived beam under discussion.

If conditions (1.7) are satisfied, then

$$\ddot{R} + \frac{\xi_0^2}{R} - \frac{\xi_0^2 R_0^2}{R^3} = 0 \tag{1.8}$$

follows from (1.6) for $R_e = R_i = R(t)$, where

$$\xi_0^2 = \frac{2e^2}{m\gamma}\left(N_i - \frac{N_e}{\gamma^2}\right) = \frac{2e^2}{M}(N_e - N_i) = \frac{2e^2 N_e}{\gamma^2}\frac{\gamma^2 - 1}{M + m\gamma}; \quad R_0 = R_{0e} = R_{0i}.$$

This equation has the steady solution $R = R_0$ and first integral

$$\dot{R}^2 + \xi_0^2 \ln\frac{R}{R_0} + \frac{\xi_0^2 R_0^2}{2R^2} = \frac{C_1^2}{2}. \tag{1.9}$$

One can construct the trajectories in phase space R, $\dot{R}$ for (1.9). For large values of the constant C_1 ($C_1 \gg \xi_0 R_0$) the phase trajectory intersects the $\dot{R} = 0$ axis at $R_1 \approx \xi_0 R_0 / C_1 <$ R_0 and $R_2 \simeq R_0 e^{\frac{C_1^2}{2\xi_0^2}} \gg R_0$ (Fig. 1). If at some time the beam has a very small radius $R_1 < R_0$, a time is found at which the radius is very large $R_2 \gg R_0$; one can convince oneself of the fact that the beam is in a state with $R \gg R_0$ for a large part of the time.

We also note the following with respect to the validity of the system of equations (1.6) describing an electron—ion beam.

The equation for the electrons or ions is satisfied exactly at those times at which the radius of the corresponding component is less than the radius of the other one. The equation for the component with the larger radius may be only valid; the more accurate the approximation, the smaller the difference in the radii — in this case an equation of motion of the form (1.3) breaks down in a small region of radii ($|R_e - R_i| \ll R_i$).

The presence of the integral of motion (1.2) shows that the boundaries of the beam reach particles having a specific value of the momentum of the transverse velocity relative to the axis. For example, the momentum should be ≈ 0 in the absence of a longitudinal magnetic field. Consequently, we have here a small fraction of the particles — the smaller it is, the smaller is the difference in radii, only a small part of whose trajectories emerge from the region characterized by the sum of the electron and ion densities. The trajectories of the bulk of the particles lie completely in this region. This fact permits using the system (1.6) in the presence of small deviations of R_e from R_i. Taking accurate account of phenomena in a narrow boundary layer of the beam goes beyond the framework of this paper.

Evidently, what has been said above is not very important in the case in which the solution for R_e and R_i is of an oscillatory nature with a relatively small amplitude $R_0 \approx R_e \approx R_i$, $|R_e - R_i| \ll R_0$.

However, the situation is altered if the difference in radii becomes sufficiently large. The equation for the component having the larger radius ceases to be valid, and consequently the validity of the entire system (1.6) breaks down.

Validity of Eqs. (1.6) for small deviations of quantities from the equilibrium solution, by which we do not necessarily have in mind in this case the steady solution with $R_e(t) \equiv R_i(t)$, is sufficient for investigation of stability problems.

2. Let us consider small perturbations of a nonsteady beam. To this end we will set $R_e = R(t) + r_1$ and $R_i = R(t) + r_2$, where $r_1, r_2 \ll R$, in Eqs. (1.7). We will assume that r_1 and r_2 do not depend upon z. Then we obtain

$$\ddot{r}_1 + \frac{\xi_1^2}{R^2} r_1 + \frac{3\xi_0^2 R_0^2}{R^4} r_1 - (\xi_0^2 - \xi_2^2)\frac{r_2}{R^2} = 0,$$
$$\ddot{r}_2 + \frac{\xi_2^2}{R^2} r_2 + \frac{3\xi_0^2 R_0^2}{R^4} r_2 - (\xi_0^2 - \xi_1^2)\frac{r_1}{R^2} = 0, \quad (2.1)$$

where

$$\xi_0^2 = 2e^2 N_e\left(\frac{2M + m\gamma(1+\gamma^2)}{m\gamma^3 (M + m\gamma)} + \frac{2}{M}\right); \quad \xi_1^2 = \frac{2e^2 N_e}{m\gamma^3}\frac{2M + m\gamma(1+\gamma^2)}{M + m\gamma};$$
$$\xi_2^2 = \frac{2e^2 N_e}{M\gamma^2}\frac{M(1+\gamma^2) + 2m\gamma^3}{M + m\gamma}.$$

One can obtain from these equations

$$\ddot{r} + \left(\frac{\xi_0^2}{R^2} + \frac{3\xi_0^2 R_0^2}{R^4}\right) r = 0 \quad (2.2)$$

for the difference $r = r_1 - r_2$. Let us investigate the stability of this equation near the steady solution $R = R_0$. Setting $R = R_0 + R_1(t)$, $R_1 \ll R_0$, we have from (1.8) the equation

$$\ddot{R}_1 + \Omega^2 R_1 = 0,$$

where $\Omega^2 = 2\xi_0^2/R_0^2$. It follows from this that $R = R_0 + a\cos(\Omega + \varphi)$, where a and φ are the amplitude and phase of oscillations of the beam radius. Now one can reduce Eq. (2.2) to the form

$$\ddot{r} + r\left(2\Omega^2 + \omega_0^2 - \frac{a}{R_0}\cos(\Omega t + \varphi)\,(7\Omega^2 + 8\omega_0^2)\right) = 0, \quad (2.3)$$

with

$$\omega_0^2 = \frac{4e^2 N_e}{R_0^2}\left(\frac{1}{m\gamma^3} + \frac{1}{M}\right).$$

It is well known that a Mathieu equation of form (2.3) with values for the coefficients of $2 + \omega_0^2/\Omega^2 = (n/2)^2$ (n is an integer, $n > 2$) describes unstable solutions (with any value of the amplitude a). This instability is a parametric resonance (see [2]). Using the expressions for Ω and ω_0 in the most interesting case ($M \gg m\gamma^3$, $\gamma \gg 1$), one can write the resonance condition in the form

$$\gamma = \left(\frac{M}{m\left(\frac{n^2}{2} - 2\right)}\right)^{1/3}. \quad (2.4)$$

Thus, parametric resonance occurs in the case of strictly determined values of the relativistic factor. This instability denotes the creation of a long-lived beam with values of γ determined by (2.4) to be impossible. We note the independence of the existence of resonance on such factors as the beam current and radius. If the condition (2.4) is not satisfied, no increase occurs in the difference of the radii of the electron and ion components of the beam, and one may expect that the beam will be stable (relative to long-lived perturbations).

3. Equations (1.6) permit a steady solution not only upon satisfaction of the conditions (1.7). One can convince oneself that in the general case the equilibrium radius of the beam R_0 should satisfy the equations

$$R_0^4+\frac{8e^2}{M\omega_{Hi}^2}(N_e-N_i)(R_0^2-R_{0i}^2)=0; \tag{3.1}$$

$$R_0^4+\frac{8e^2}{m\gamma\omega_{He}^2}\left(N_i-\frac{N_e}{\gamma^2}\right)(R_0^2-R_{0e}^2)=0. \tag{3.2}$$

If one sets $\omega_{Hi}=\omega_{He}=0$ and R_{0i} and $R_{0e}=R_0$ as in Sec. 2, but assumes, however, the perturbations to depend on z, i.e., sets $r_{1,2}=s_{1,2}e^{ikz}$, and takes account of the fact that one should express the total derivative with respect to the time as $\partial/\partial t+ik\beta_0c$ for the electron component, then in the most interesting case ($M\gg m\gamma^3$, $\gamma\gg1$) one can obtain from (2.1)

$$-(\omega-kv)^2s_1+\omega_e^2s_1-\omega_i^2s_2=0,$$
$$-\omega^2s_2+2\omega_i^2s_2-\omega_e^2s_1=0,$$

where

$$\omega_e^2=\frac{4e^2N_e}{m\gamma^3R_0^2};\quad \omega_i^2=\frac{4e^2N_e}{MR_0^2};\quad v=\beta_0c.$$

The dispersion equation is of the form

$$(\omega^2-2\omega_i^2)((\omega-kv)^2-\omega_e^2)=\omega_i^2\omega_e^2.$$

A plot of the function $\Phi(\omega)=(\omega^2-2\omega_i^2)((\omega-kv)^2-\omega_e^2)$ for different values of the wavenumber k is given in Fig. 2. If $(kv)^2<\omega_e^2$ or $kv\gg\omega_e$ (Figs. 2a and b), then the dispersion equation has four different real roots, which indicates stability of the system relative to long-lived and short-lived perturbations. In the case $kv\approx\omega_e$ (Fig. 2a) there exist two complex-conjugate roots, one of which corresponds to an increase of the perturbations. With $\omega_e\approx kv$ we get $(\omega^2-2\omega_i^2)\omega(\omega-2\omega_e)=\omega_i^2\omega_e^2$, an approximate solution in the region $\omega_e\gg\omega\gg\omega_i\omega^3=-\omega_i^2\omega_e/2$. From this follows the growth increment $\sim(\sqrt{3}/2^{1/3})\omega_i^{2/3}\omega_e^{1/3}$.

Thus, a steady beam is unstable relative to axisymmetric perturbations with the wave number $k=\omega_e/v=\sqrt{4e^2N_e/m\gamma^3R_0^2v^2}$. The width of the instability shell $\Delta k\sim\omega_i/v$. We note that not too lengthy a beam (with fixed ends), whose length l satisfies the condition $l<1/k$, may be stable.

4. It is convenient for classification of the equilibrium states of a beam having a magnetic field to introduce the parameter η with the help of the equation

$$\eta=\frac{8e^2}{M\omega_{Hi}^2}\frac{N_e-N_i}{R_{0i}^2}>0.$$

First we assume that the action of the magnetic field on the ions is small, i.e., $\eta\gg1$. Then

$$R_0=R_{0i},\ R_{0i}^4\frac{m\gamma\omega_{He}^2}{8e^2\left(N_i-\frac{N_e}{\gamma^2}\right)}+R_{0i}^2=R_{0e}^2 \tag{4.1}$$

follows from (3.1) and (3.2). Thus, equilibrium is possible only in the case in which the emittances of the ion and electron components of the beam satisfy Eq. (4.1).

One can derive

$$\ddot r_1+r_1\left[\frac{8e^2}{m\gamma}\frac{N_i}{R_0^2}-\frac{4e^2}{m\gamma^3}\frac{N_e}{R_0^2}\right]+\omega_{He}^2r_1-\frac{4e^2N_i}{m\gamma R_0^2}r_2=0,$$

$$\ddot r_2+r_2\left[\frac{8e^2N_e}{MR_0^2}-\frac{4e^2N_i}{MR_0^2}\right]-\frac{4e^2N_e}{MR_0^2}r_1=0$$

in place of (2.1). If one assumes that $N_e\gg N_i\gg N_e/\gamma^2$ and introduces the parameter $\xi=MN_i/m\gamma N_e$, then one can represent the dispersion equation in the form

$$((\omega-kv)^2-2\omega_i^2\xi-4\omega_{He}^2)(\omega^2-2\omega_i^2)=\omega_i^4\xi.$$

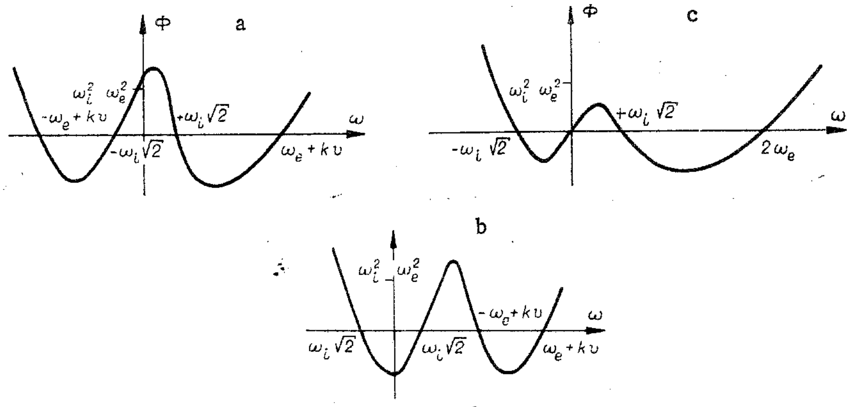

Fig. 2

This equation has four different real roots if $\omega_{He}^2 \gg \omega_i^2\xi$. Thus, in a sufficiently strong magnetic field the equilibrium state of a beam with magnetized ions will be stable with respect to axisymmetric perturbations.

We obtain from (3.1) and (3.2) upon consideration of the opposite case ($\eta \ll 1$) $R_0 = \sqrt{\eta}R_{0i}$, and $R_{0e}^2 = m\gamma\omega_{He}^2\eta^2R_{0i}^4/(8e^2(N_i - N_e/\gamma^2)) + \eta R_{0i}^2$. The equations for the deviations of the radii of the electron and ion components from the equilibrium value are of the form

$$\ddot{r}_1 + \left(\frac{8e^2N_i}{m\gamma R_0^2} - \frac{4e^2N_e}{m\gamma^3R_0^2}\right)r_1 + \omega_{He}^2r_1 - \frac{4e^2N_i}{m\gamma R_0^2}r_2 = 0,$$

$$\ddot{r}_2 + \frac{8e^2}{M}\frac{N_e + N_i}{R_0^2}r_2 - \frac{4e^2N_e}{MR_0^2}r_1 + \omega_{Hi}^2r_2 = 0,$$

which results in the dispersion equation

$$\left((\omega - kv)^2 - 2\omega_i\xi - \omega_{He}^2\right)\left(\omega^2 - \frac{\omega_i^2}{2} - \omega_{Hi}^2\right) = \xi\omega_i^4.$$

As above, satisfaction of the condition $\omega_{Hi}^2 \gg \omega_i^2\xi$ is sufficient in order that this equation have four real roots. This indicates that equilibrium of a beam with magnetized ions in a strong magnetic field will also be stable relative to axisymmetric perturbations.

The problem of the stability of an electron–ion beam relative to axisymmetric perturbations has also been investigated in [6, 7]. The mechanism of "focused instability" [6] corresponds overall to the instability which we have discussed of a beam without a magnetic field, however, under several other conditions (we have investigated a beam with nonzero emittance, which may in the general case be different for the electron and ion components).

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